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NEW STOCHASTIC DOMINANCE THEORY FOR INVESTORS WITH RISK-AVERSE AND RISK-SEEKING UTILITIES WITH APPLICATIONS INCLUDING SOLUTIONS FOR THE FRIEDMAN-SAVAGE PARADOX AND THE DIVERSIFICATION PUZZLE

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New stochastic dominance theory for investors with risk-averse and risk-seeking utilities with applications including solutions for the Friedman-Savage paradox and the diversification puzzle *

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Abstract

In this paper, we first state some well-known problems including the Friedman-Savage paradox raised by Friedman and Savage (1948) who wonder why individuals would like to buy insurance as well as buy lottery tickets. To provide solutions to the problems, we first use the idea from Fishburn and Kochenberger (1979), Thon and Thorlund-Petersen (1988), and Chew and Tan (2005) to use two-way stochastic dominance to define the j-order risk-averse and risk-seeking utility that consists of both risk-averse and risk-seeking components and we call the utility AD utility and call investors with AD utility AD investors. Thereafter, we develop a new stochastic dominance theory for AD investors and we call the theory ADSD theory. We then develop some properties for the ADSD theory, including properties of expected-utility maximization, hierarchy, transitivity, and diversification, and properties under the additional condition of equal mean so that we can use the theory to get the solutions for all the problems and hypotheses we set in this paper. Applying the ADSD theory, we first get a new solution for the Friedman-Savage paradox. In addition, we find that AD investors could invest in both completely diversified portfolio and individual assets and, in general, buy any pair of both less-risky and more-risky assets. For example, AD investors could invest in both bonds and stocks, both bonds and futures, and both stocks and futures to get higher expected utility.

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1 Introduction

The fundamental assumption in standard theories of asset valuation, lottery choice, and others is risk aversion (Bernoulli, 1738). However, it is well known that using risk aversion cannot solve the paradox raised by Friedman and Savage because individuals who like to buy lottery tickets are not risk-averse. Some studies, for example, Pratt (1964), have found that some individuals are risk-seeking. Could risk-seeking solve the paradox raised by Friedman and Savage? The answer is no because individuals who like to buy insurance are not risk-seeking.

In this paper, we aim to provide a solution to the Friedman-Savage paradox raised by Friedman and Savage (1948) who wonder why individuals would like to buy insurance as well as buy lottery tickets. We also raised other important related problems and discussed them in Section 3. To provide solutions to the problems, we first modify the two-piece utility function idea proposed by Fishburn and Kochenberger (1979) and others and use the idea proposed by Thon and Thorlund-Petersen (1988) to use two-way stochastic dominance to first define the j-order risk-averse and risk-seeking (AD) utility[1].

Thereafter, we develop a new stochastic dominance (SD) theory for investors with AD utility (We call them AD investors) that consists of both risk-averse and risk-seeking components. We then develop some properties for the ADSD theory. Applying the ADSD theory developed in our paper, we provide a new solution for the Friedman-Savage paradox and the solutions for all other problems and hypotheses we set in this paper.

[1]We call this AD utility. Readers may refer to Definition 4.2 for the definition of AD utility and the reason why we call it AD utility.

The rest of the paper is organized as follows. Section 2 discusses some literature related to our paper and Section 3 states all the problems and hypotheses we planned to solve in this paper. Section 4 states the motivations of our study and presents all the definitions and notations used in our paper and in Section 5, we develop SD theory and some properties of the SD theory. In Section 6, we discuss how to apply the theory we developed in our paper to get the solutions for all the problems and hypotheses we set in this paper. Section 7 concludes, discusses the limitations of the theory we developed in our paper, and suggests further extensions of the theory.

2 Literature Review

Bernoulli (1738), Pratt (1964), Arrow (1965), and many others assume subjects are risk-averse, and many fundamental theories are developed based on the assumption of risk aversion. For example, Quirk and Saposnik (1962), Fishburn (1964), Hadar and Russell (1969), Hanoch and Levy (1969), and many others have developed the theory of Stochastic Dominance (SD) for risk averters. There are many interesting applications of using the SD for risk averters, see, for example, Chan, *et al.* (2012), Fong, *et al.* (2005), Gasbarro, *et al.* (2007), Lean, *et al.* (2007, 2012), Wong, *et al.* (2008, 2018), Qiao and Wong (2015), Hoang, *et al.* (2015, 2019), and many others.

In reality, individuals may not be risk averse. Tobin (1958), Slovic (1964), Dyer and Sarin (1982), Machina (1982, 1985, 2001), Von Winterfeldt and Edwards (1986), MacCrimmon and Wehrung (1986, 1990), Bromiley and Curley (1992), Sarin and Weber (1993), Schoemaker (1993), Wärneryd (1996), Weber and Milliman (1997), Isaac and James (2000), Loewenstein, *et al.* (2001), Holt and Laury (2002), Lejuez *et al.* (2002), Weber, *et al.* (2002), Blais and Betz (2002), Wallsten, *et al.* (2005), Blais and Weber (2006), Figner, *et al.* (2009), Figner and Weber (2011), Di Rago, *et al.* (2012), Chan, *et al.* (2019), have found that some individuals are risk-seeking. Based on the assumption of risk seeking, Hammond (1974), Meyer (1977), Hershey and Schoemaker (1980), Stoyan

(1983), Li and Wong (1999), Wong and Li (1999), Wong (2006, 2007), Levy (2015), Guo and Wong (2016), Chan, *et al.* (2019), Bai, *et al.* (2020), and many others have developed the SD theory for risk seekers. There are many applications by using the SD theory for both risk averters and risk seekers, see, for example, Qiao, *et al.* (2012, 2013), Lean, *et al.* (2015), Hoang, *et al.* (2015a, 2018), and many others.

There are some other works aim to develop some theories to explain the Friedman-Savage paradox, see, for example, Markowitz (1952a), Ng (1965), Williams (1966), Fishburn and Kochenberger (1979), Machina (1982, 2001), Shefrin and Statman (1993), Benartzi and Thaler (1995), Myagkov and Plott (1997), Levy and Wiener (1998), Levy and Levy (2002, 2004), Pennings and Smidts (2003), Wang and Fischbeck (2004), Gneezy, *et al.* (2006), Wong and Chan (2008), Broll, *et al.* (2010) and many others. For instance, Markowitz (1952a) proposes to use a utility function that contains both convex and concave portions in both negative and positive domains. Thon and Thorlund-Petersen (1988) propose to use two-way stochastic dominance as the ordering corresponding to the unanimous ranking given by all risk-averse and risk-loving agents. In addition, assuming that consumers have constant absolute risk attitudes which are strictly averse to small as well as symmetric risks, and display longshot preference behavior, Chew and Tan (2005) derive equilibrium demands for fixed-prize and variable-prize sweepstakes and determine the profit-maximizing prize level and pay-out ratio, respectively. Kahneman and Tversky (1979), Tversky and Kahneman (1992), Benartzi and Thaler (1995), Levy and Wiener (1998), Levy and Levy (2002, 2004), Wong and Chan (2008), and others develop the theory of S-shaped utility function that is concave in the positive domain but convex in the negative domains and some academics expect that the theory can be used to explain the Friedman-Savage paradox. On the other hand, Thaler and Johnson (1990) and others have observed behaviors of investors having reversed S-shaped utility functions that are convex in the positive domain but concave in the negative domains and Levy (2002, 2004), Wong and Chan (2008), and others have developed the stochastic dominance theory for investors with reversed S-shaped utility functions. There are many applications by using the SD theory for investors with (reversed) S-shaped utility functions, see, for example, Fong, *et al.* (2008), Clark, *et al.* (2016), and many others, and some academics expect that the theory can be used to explain the Friedman-Savage paradox. Nonetheless, both S-shaped and reverse S-shaped utility function cannot be used to solve the Friedman-Savage paradox.

3 Problems and Hypotheses

In this paper, we first consider the following problem:

Problem 1 *Consider the pair $\vec{X} = (X_1, X_2)'$ of two assets: one asset, X_1, is less risky while the other asset, X_2, is riskier, will an investor buy both assets?*

To explain this puzzle, different academics have different ideas. For example, Markowitz (1952) proposes a utility function that consists of both convex and concave regions in both positive and negative domains. Ng (1965), Machina (1982), Gneezy, *et al.* (2006), and others provide different explanations to the puzzle. In this paper, we propose a new idea to handle this problem. To do so, we modify the idea of using a two-piece utility function proposed by Fishburn and Kochenberger (1979) and others to first define the j-order risk-averse and risk-seeking (AD) utility $\vec{u}_j = (u_{1j}, u_{2j})'$ as stated in Definition 4.2 that consisting both j-order risk-averse ($u_{1j} \in U_j^A$) and j-order risk-seeking ($u_{2j} \in U_j^D$) components such that investors with AD utility will use u_{1j} to evaluate the less-risky assets X_1 and use u_{2j} to evaluate the more-risky asset X_2. We will explain the solutions of Problem 1 in Section 6 by using the theory developed in this paper.

In this paper, we also consider the following problem:

Problem 2 *For any two pairs of assets $\vec{X} = (X_1, X_2)'$ and $\vec{Y} = (Y_1, Y_2)'$ in which both X_1 and Y_1 are less risky and X_2 and Y_2 are riskier. Which pair of assets will investors with AD utility buy?*

To provide an answer to Problem 2, in this paper, we develop a new stochastic dominance (SD) theory as stated in Definition 4.4 for investors with AD utility. We call investors with AD utility **AD investors** and call the theory *ascending-and-descending (AD) SD theory* for AD investors, in short, *ADSD theory* because it consists of two parts: ascending SD (ASD) for risk averters and descending SD (DSD) for risk seekers. Readers may refer to Sriboonchitta, *et al.* (2009), Levy (2015), Guo and Wong (2016), Chan *et al.* (2019), Bai, *et al.* (2020), and others for more information on the ascending SD theory for risk averters and descending SD theory for risk seekers. In the next section, we will show that the preference for different pairs of less-risky and more-risky assets is equivalent to the expected utility maximization of the assets for AD investors. We will also develop other properties of the ADSD theory that can be used to solve all the problems stated in the next section. We will explain the solutions of Problem 2 in Section 6 by using the theory developed in this paper.

Problem 2 states a general case. One can consider some more special cases. For example, we study the following problem to address the Friedman-Savage paradox that subjects could buy insurance and try their luck with lotteries as well.

Problem 3 (Friedman-Savage paradox) *Is there any investor who will buy both insurance and try their luck with lotteries to get a higher expected utility?*

The ADSD theory developed in this paper provides a good insight to get a solution to the problem. Based on the theory, we find that AD investors will buy both insurance and try their luck with lotteries to get higher expected utility as discussed in Section 6.

Based on the modern finance theory developed by Markowitz (1952b) and others, many fund managers recommend investors buy index funds, or in other words, buy a completely diversified portfolio if the returns of all assets are iid (Egozcue and Wong, 2010). However, other fund managers may recommend investors not buy the index funds, but buy one or a few individual assets (Statman, 2004; Egozcue, *et al.,* 2011; Lozza, *et al.,* 2018).

This phenomenon is called the diversification puzzle (Statman, 2004). We note that fund managers are not necessarily benevolent individuals, and therefore, they have their own agenda, which is generally focused on maximizing the fees collected from their clients, therefore, agency problems are always present in this type of recommendation. To get solutions for the suggestions, we suggest to study the following problem:

Problem 4 (diversification puzzle) *Is there any investor who prefers not to buy one stock as well as not buy too many stocks but buy a few stocks to get higher expected utility under some conditions?*

The theory developed in this paper provides a good insight into solving the above problems as discussed in the next section. Based on the ADSD theory we developed in this paper, we find that AD investors prefer not to buy one stock as well as do not buy too many stocks but buy a few stocks to get higher expected utility. We will explain the solutions in Section 6 by using the theory developed in this paper.

4 Motivations, Definitions, and Notations

For Problem 1, we conclude that it is possible that subjects could buy two assets with different risk levels because Friedman and Savage (1948) and others have observed that subjects could buy assets with different risk levels, for example, subjects could try their luck with lotteries as well. Markowitz (1952), Gneezy, *et al.* (2006), and others explain the puzzle. In this paper, we extend the work of studying the paradox by finding a new solution to the paradox. To do so, we first make the following definition:

Definition 4.1 *Suppose the standard deviation of* X_i *is* σ_i *for* $i = 1, 2$ *such that* $\sigma_1 \leq \sigma_2$*. Then, for this pair of assets* $\vec{X} = (X_1, X_2)'$*, we say* X_1 *is less risky and* X_2 *is riskier.*

For the pair of assets $\vec{X} = (X_1, X_2)'$ and for any integer j, we are now ready to define investor with AD utility $\vec{U}_j^{AD} = (u_{1j}, u_{2j})'$ that consists of a risk-averse component u_{1j} and a risk-seeking component u_{2j} to satisfy the following definition:

Definition 4.2 *For the pair of assets $\vec{X} = (X_1, X_2)'$ defined in Definition 4.1 and for any integer j, $\vec{u}_j = (u_{1j}, u_{2j})' \in \vec{U}_j^{AD} = (U_j^A, U_j^D)$ is an AD utility function for $\vec{X} = (X_1, X_2)'$ satisfying*

$$\vec{u}(\vec{X}) = u_{1j}(X_1) + u_{1j}(X_2) \tag{4.1}$$

such that $u_{1j} \in U_j^A$ and $u_{2j} \in U_j^D$ with expectation

$$E\big[\vec{U}_j^{AD}(\vec{X})\big] = E\big[u_{1j}(X_1)\big] + E\big[u_{2j}(X_2)\big] \ , \tag{4.2}$$

in which U_j^A and U_j^D are sets of utility functions u such that $U_j^A = \{u : (-1)^i u^{(i)} \leq 0 \, , i = 1, \cdots, j\}$, $U_j^D = \{u : u^{(i)} \geq 0 \ , \ i = 1, \cdots, j\}$, and $u^{(i)}$ is the i^{th} derivative of the utility function u.

We note that in Definition 4.2, the theory can easily be extended to include non-differentiable utility (Wong and Ma, 2008). In Definition 4.2, we assume the utility $\vec{u}(\vec{X})$ possesses a separability property such that the two pieces of utilities $u_{1j}(X_1)$ and $u_{1j}(X_2)$ and their expectation utilities as shown in Equations (4.1) and (4.2). We note that it is common to impose the assumption of the separability property (Keeney and Raiffa, 1976). In addition, we assume X_1 and X_2 are independent in Assumption 4.1.

We call $u_{1j} \in U_j^A$ the j-order risk-averse component and $u_{2j} \in U_j^D$ the j-order risk-seeking component of the utility $\vec{U}_j^{AD}$ and we call the investor with utility $\vec{U}_j^{AD} = (u_{1j}, u_{2j})'$ the **j-order AD investor with a j-order AD utility** that consisting of both j-order risk-averse (u_{1j}) and j-order risk-seeking (u_{2j}) components or simply call them the **(j-order) AD investors**.[2] Without loss of generality, we assume that X_1 and X_2 satisfy the following assumption[3] so that the expressions in both (4.1) and (4.2) are well-defined:

Assumption 4.1 *X_1 and X_2 defined in Definition 4.1 are independent and investors put equal weights of their wealth to invest in X_1 and X_2.*

[2] We call this utility the j-order AD utility because the j-order risk-averse utility is related to ascending SD (ASD) for risk averters and the j-order risk-seeking utility is related to descending SD (DSD) for risk seekers.

[3] One could relax the "equal-weight" assumption easily. For "independence" assumption, we leave this to future research.

If there are n assets $\vec{X}_n = (X_1, \cdots, X_n)'$, a portfolio of $\vec{X}_n$ without short selling is defined by a convex combination, $\overrightarrow{\lambda_n}'\overrightarrow{X}_n$, of the n assets $\vec{X}_n$ for any $\overrightarrow{\lambda_n} \in S_n^0$ (S_n) where

$$
\begin{aligned}
S_n^0 &= \left\{ (s_1, s_2, \cdots, s_n)' \in \mathbb{R}^n : \ 0 \le s_i \le 1 \text{ for any } i \ , \ \sum_{i=1}^{n} s_i = 1 \right\} , \\
S_n &= \left\{ (s_1, s_2, \cdots, s_n)' \in \mathbb{R}^n : 1 \ge s_1 \ge s_2 \ge \cdots \ge s_n \ge 0, \sum_{i=1}^{n} s_i = 1 \right\} , \qquad (4.3)
\end{aligned}
$$

in which $\mathbb{R}$ is the set of real numbers and the i^{th} element of $\overrightarrow{\lambda_n}$ is the weight of the portfolio allocation on the i^{th} asset of return X_i. A portfolio will be equivalent to return on asset i, which we call a *specialized portfolio*, or simply a *specialized asset*, or an *individual asset*, if $s_i = 1$ and $s_j = 0$ for all $j \neq i$. It is a *partially diversified portfolio* if there exists i such that $0 < s_i < 1$ and is the *completely diversified portfolio* if $s_i = \frac{1}{n}$ for all $i = 1, 2, \cdots, n$. We note that we include the condition of $\sum_{i=1}^{n} s_i = 1$ is only for convenience. All the results developed here work well without this condition. We then follow Hardy, et al. (1934), Egozcue and Wong (2010), Guo and Wong (2016), and others to state the following definition to order the elements in S_n:

Definition 4.3 *Let $\vec{\alpha}_n, \vec{\beta}_n \in S_n$ in which S_n is defined in (4.3). $\vec{\beta}_n$ is said to majorize $\vec{\alpha}_n$, denoted by $\vec{\beta}_n \succeq_M \vec{\alpha}_n$, if $\sum\limits_{i=1}^{k} \beta_i \ge \sum\limits_{i=1}^{k} \alpha_i$, for all $k = 1, 2, \cdots, n$.*

An individual prefers X to Y, or equivalently, prefer F to G (von Neumann and Morgenstern, 1944) if

$$
\Delta E u \equiv E\big[u(X)\big] - E\big[u(Y)\big] \ge 0, \qquad (4.4)
$$

where $E\big[u(X)\big] \equiv \int_a^b u(x)dF(x)$ and $E\big[u(Y)\big] \equiv \int_a^b u(x)dG(x)$.

In this paper, we consider random variables, X and Y, defined on $[a, b]$ together with their corresponding cumulative distribution functions (CDFs) F and G, and their corresponding probability density functions (PDFs) f and g, respectively. The following notations will be used throughout this paper (Eckern, 1980):

$$
\mu_F = \mu_X = E(X) = \int_a^b t \, d\, F(t) \quad , \quad \mu_G = \mu_Y = E(Y) = \int_a^b t \, d\, G(t) \quad , \qquad (4.5)
$$

$$
h(x) = H_0^A(x) = H_0^D(x) \quad , \quad H_j^A(x) = \int_a^x H_{j-1}^A(y)\, dy \quad , \quad H_j^D(x) = \int_x^b H_{j-1}^D(y)\, dy \ ,
$$

where $h = f$ or g and $H = F$ or G for any $j > 1$.[4] In (4.5), $\mu_F = \mu_X$ is the mean of X and $\mu_G = \mu_Y$ is the mean of Y.

To provide solutions to Problems 2 and 3, we define a new stochastic dominance theory for AD investors in the following:

Definition 4.4 *For any integer $j > 0$ and given two pairs of assets $\vec{X} = (X_1, X_2)'$ and $\vec{Y} = (Y_1, Y_2)'$ defined in Definition 4.1 in which both X_1 and Y_1 are less risky and both X_2 and Y_2 are riskier, $\vec{X}$ is at least as good as $\vec{Y}$ in the sense of jADSD, denoted by $\vec{X} \succeq_j^{AD} \vec{Y}$, if and only if $X_1 \succeq_j^{A} Y_1$ and $X_2 \succeq_j^{D} Y_2$, where jADSD stands for j-order ADSD.*[5]*.*

We call the SD theory for the AD investors **AD stochastic dominance theory** or in short **ADSD theory** denoted by ADSD because it consists of two parts: ascending SD (ASD) for risk averters and descending SD (DSD) for risk seekers.

5 Theory

We now develop some properties for the ADSD theory that can be used to obtain solutions to the problems we set in our paper. We first show the relationship between the dominance relationship on each component of each pair of assets and the dominance relationship of the entire pairs of assets as shown in the following theorem:

Theorem 5.1 *For any two pairs of assets $\vec{X} = (X_1, X_2)'$ and $\vec{Y} = (Y_1, Y_2)'$ defined in Definition 4.1 in which both X_1 and Y_1 are less risky and both X_2 and Y_2 are riskier, if $X_1 \succeq_i^{A} Y_1$ and $X_2 \succeq_j^{D} Y_2$, then $\vec{X} \succeq_k^{AD} \vec{Y}$ with $k = \max\{i, j\}$ for any pair of integers $\{i, j\}$.*

Proof. Refer to Wong, Ma, Qiao, Broll, and Vieito (2025). ■

[4] The above definitions are commonly used in the literature; see for example, Wong and Li (1999), Wong (2007), Guo and Wong (2016), and Chan, *et al.* (2019).

[5] We note that one could define $\vec{X} \succ_j^{AD} \vec{Y}$ if and only if $X_1 \succ_j^{A} Y_1$ and $X_2 \succ_j^{D} Y_2$ for strictly j-order ADSD. Without loss of generality, we assume strictly for all the ADSD being discussed in our paper but we use the notation "$\succeq_j^{AD}$" instead of "$\succ_j^{AD}$". Readers may refer to Levy (2015), Guo and Wong (2016) and the references therein for the definition of the ASD and DSD and the notations $\succeq_j^{A}$ and $\succeq_j^{D}$.

The most important part of the ADSD theory that can be used to obtain solutions to Problems 2 and 3 is the diversification property. For any n assets $(X_1, \cdots, X_n)'$, a portfolio $\sum_{i=1}^n \lambda_i X_i$ is the *completely diversified portfolio* if $\lambda_i = \frac{1}{n}$ for any $i = 1, 2, \cdots, n$; a *specialized portfolio*, or simply a *specialized asset*, if $\lambda_i = 1$ and $\lambda_i = 0$ for all $j \neq i$; and a *partially diversified portfolio* if there exists i such that $0 < \lambda_i < 1$. Hadar and Russell (1971), Tesfatsion (1976), and others have studied the diversification behavior for risk averters while Li and Wong (1999), Wong (2007), Guo and Wong (2016), and others have studied the diversification behavior for risk seekers on completely diversified portfolio, partially diversified portfolio, and specialized asset. On the other hand, Egozcue and Wong (2010) develop a theory to compare the preferences of different convex combinations of assets for risk averters while Guo and Wong (2016) extend the theory to compare the preferences of different convex combinations of assets for risk seekers. Considering n independent and identically distributed (iid) pairs of assets $\vec{X}_i = (X_{1i}, X_{2i})'$ for $i = 1, \cdots, n$, we extend their work by developing the theorem to examine the preferences of the AD investors among completely diversified portfolio, partially diversified portfolio, and specialized asset:

Theorem 5.2 *Consider n iid pairs of assets $\vec{X}_i = (X_{1i}, X_{2i})'$ for $i = 1, \cdots, n$,*

$$\left(\frac{1}{n}\sum_{i=1}^{n} X_{1,i}, X_{2,i}\right) \succeq_2^{AD} \left(\sum_{i=1}^{n} \alpha_i X_{1,i}, \sum_{i=1}^{n} \beta_i X_{2,i}\right) \succeq_2^{AD} \left(X_{1,i}, \frac{1}{n}\sum_{i=1}^{n} X_{2,i}\right)$$

for any $0 < \alpha_i < 1$ and $0 < \beta_i < 1$ with $1 \leq i \leq n$, $\sum_{i=1}^n \alpha_i = 1$, and $\sum_{i=1}^n \beta_i = 1$.

Proof. Refer to Wong, Ma, Qiao, Broll, and Vieito (2025). ■

We note that in this paper, when we talk about n pairs of assets $\vec{X}_i = (X_{1i}, X_{2i})'$, it does not mean that the number of X_{1i} must be the same of the number of X_{2i} and one must choose X_{1i} to match with X_{2i}, cannot choose X_{1i} to match with X_{2j} for any $i \neq j$. We use this notation only for convenient purpose. In fact, there could be m $\{X_{1i}\}$ and n $\{X_{2j}\}$ so that one could choose any X_{1i} from $\{X_{1i}\}$ and any X_{2j} from $\{X_{2j}\}$ so that there are actually mn pairs of $\vec{X}_i = (X_{1i}, X_{2i})'$. Now, let $X_{1i} = X_{2i}$ for any $i = 1, \cdots, n$, from Theorem 5.2, we obtain the following theorem:

Theorem 5.3 *For a vector of any n iid assets $\vec{X}_n = (X_1, \cdots, X_n)'$,*

$$\left(\overrightarrow{\left(\frac{1}{n}\right)}'\overrightarrow{X}_n, X_i\right) \succeq_2^{AD} \left(\overrightarrow{\alpha_n}'\overrightarrow{X}_n, \overrightarrow{\beta_n}'\overrightarrow{X}_n\right) \ ,$$

for any $\overrightarrow{\alpha_n} = (\alpha_1, \cdots, \alpha_n)'$, $\overrightarrow{\beta_n} = (\beta_1, \cdots, \beta_n)'$, and $\overrightarrow{\left(\frac{1}{n}\right)} = \left(\frac{1}{n}, \cdots, \frac{1}{n}\right)' \in S_n^0$.

Proof. Refer to Wong, Ma, Qiao, Broll, and Vieito (2025). ■

The findings in both Theorems 5.2 and 5.3 are interesting but the former can only allow investors to compare any pair from an individual asset, a partially diversified portfolio, and the completely diversified portfolio while the latter can only allow investors to compare a partially diversified portfolio and the completely diversified portfolio, both theorems cannot be used to compare any pair of partially diversified portfolios. To circumvent the limitation, in this paper we develop the following theorem:

Theorem 5.4 *Consider n $(n > 1)$ assets X_i $(i = 1, \cdots, n)$ with the same mean such that $X_i \succeq_2^A X_j$ for any $1 \leq i \leq j \leq n$,*

$$(X_1, X_n) \succeq_2^{AD} (X_i, X_j)$$

for any $1 \leq i, j \leq n$.

Proof. Refer to Wong, Ma, Qiao, Broll, and Vieito (2025). ■

Applying Theorem 5.4, we get the following theorem:

Theorem 5.5 *For any asset X_i $(i = 1, 2, 3)$ with the same mean such that $X_i \succeq_2^A X_j$ for any $1 \leq i \leq j \leq 3$, then we have*

$$(X_1, X_3) \succeq_2^{AD} (X_i, X_j)$$

for any $1 \leq i, j \leq 3$.

We turn to consider $n = 2$ in Theorem 5.4 with assets X_i $(i = 1, 2)$ with the same mean such that $X_1 \succeq_2^A X_2$, for example, X_1 is the return of a bond and X_2 is the return of a stock, then from Theorem 5.4, we obtain the following corollary to provide an answers the problems we set in our paper.

Theorem 5.6 *Consider X_1 and X_2 with the same mean such that $X_1 \succeq_2^A X_2$,*

$$(X_1, X_2) \succeq_2^{AD} (X_i, X_i)$$

for any $i = 1, 2$.

6 Applications and Further Studies

In this section, we apply the ADSD theory to provide a solution for each of the problems we set in Section 3. We also discuss some directions for further studies in this section.

6.1 Will investors buy both less-risky and riskier assets?

We first discuss a solution to Problem 1 by using the ADSD theory developed in this paper. To solve Problem 1, we assume that there are investors with p-order AD utility $\vec{u}_p = (u_{1p}, u_{2p})'$ as stated in Definition 4.2 and we assume that the AD investors are considering n pairs of assets $\vec{X}_q = (X_{1,q}, X_{2,q})'$ to invest in which $X_{1,q}$ is less risky and $X_{2,q}$ is riskier with $(X_{1,q}, X_{2,q})$ satisfying Assumption 4.1 for any q. We assume that among $X_{1,q}$, there is one asset, say $X_{1,*}$ such that $X_{1,*} \succeq_i^A X_{1,p}$ for any $p \neq *$ and we assume that among $X_{2,q}$, there is one asset, say $X_{2,*}$ such that $X_{2,*} \succeq_i^D X_{2,p}$ for any $p \neq *$. Thereafter, applying Theorem 5.1, we conclude that $\vec{X}_* = (X_{1,*}, X_{2,*}) \succeq_k^{AD} \vec{X}_q = (X_{1,q}, X_{2,q})$ with $k = \max\{i, j\}$ and for any $p \neq *$ for any pair of integers $\{i, j\}$ and any AD investors will invest in $\vec{X}_*$.

We turn to discuss a solution to Problem 2 by using the ADSD theory developed in this paper. To do so, one can apply Theorem 5.4 to provide an answer in the way that that for any n assets X_i $(i = 1, \cdots, n)$ with equal mean such that $X_i \succeq_2^A X_j$ for any

$1 \leq i \leq j \leq n$, different from risk averter who prefers to buy only the least risky asset and risk seeker who prefers to buy only the riskiest asset, an AD investor with will buy both the least risky and the riskiest assets.

6.2 The Friedman-Savage paradox

To provide a solution to the Friedman-Savage paradox as stated in Problem 3 by using the ADSD theory developed in this paper, one can consider X_1 be the return of buying insurance and X_2 be the return of buying a lottery, then applying Theorem 5.6, one could consider that any second-order AD investor will prefer to invest in (X_1, X_2) to either X_1 or X_2, this means that a second-order AD investor will prefer to buy both insurance and lottery to buy only insurance or only lottery to get higher expected utility.

6.3 The diversification puzzle

Last, we discuss a solution to Problem 4 by using the ADSD theory developed in this paper. To do so, we assume that in the market there is at least one AD investor who can only consider to investing in the following portfolios/assets: $\sum_{i=1}^{n} \alpha_i X_{1,i}$, $\sum_{i=1}^{n} \beta_i X_{2,i}$, $X_{1,i}$, and $\frac{1}{n}\sum_{i=1}^{n} X_{2,i}$ but not $\frac{1}{n}\sum_{i=1}^{n} X_{1,i}$ and $X_{2,i}$ as stated in Theorem 5.2. one may wonder why an investor who can only consider to investing in the following portfolios/assets: a) $\sum_{i=1}^{n} \alpha_i X_{1,i}$, b) $\sum_{i=1}^{n} \beta_i X_{2,i}$, c) $X_{1,i}$, and d) $\frac{1}{n}\sum_{i=1}^{n} X_{2,i}$ but not e) $\frac{1}{n}\sum_{i=1}^{n} X_{1,i}$ and f) $X_{2,i}$ as stated in Theorem 5.2. we note that this is possible. For example, n could be too big that it is impossible for many small investors to get (e) $\frac{1}{n}\sum_{i=1}^{n} X_{1,i}$. Also, the price of (f) $X_{2,i}$ could be too high that it is impossible for most, if not all, small investors to get (f). Thus, it is completely possible that an investor will not consider investing in (e) and (f) but then one may wonder why it is possible that the same investor could invest in (a), (b), (c), and (d). This is possible because first, the price of $X_{2,i}$ could be too high but not $X_{1,i}$ whose price is reasonably low that most investors can afford to invest. Second, there could have some fund companies selling (a), (b), and (c) as products at low prices so that small investors can afford to buy. Thus, it is completely possible that some investors can invest in (a), (b), (c), and (d) but not (e) and (f). Now, one is ready to apply Theorem 5.2 to get the solution for Problem 4 that there are some AD investors who prefer to invest in a partially diversified portfolio to both an individual asset and a completely diversified portfolio.

6.4 Could the ADSD theory be compared with the theories for investors with S-shaped utilities?

Before we discuss the issue, we first discuss some possible extensions by using the ADSD theory developed in this paper. Below is the first one:

Extension 6.1 *Equation* (4.1) *can be extended to*

$$\vec{u}(\vec{X}) = \omega_1 u_{1j}(X_1) + \omega_2 u_{2j}(X_2) \ , \tag{6.1}$$

where ω_1 *and* ω_2 *may not be constants.*

We note that in Equation (4.1), we assume that $\omega_1 = \omega_2 = 1$. However, in Extension 6.1, we not only relax the restriction with $\omega_1 = \omega_2 = 1$ so that $\omega_1 \neq \omega_2$, but also include the situation in which ω_1 and ω_2 may not be constants. They not only could be varied but also could vary on time so that $\vec{u}(\vec{X})$ in Equation (6.1) could vary on time.

Remark 6.1 *One could further extend the theory by relaxing the independence assumption in Extension 6.1. We note that in some empirical analysis,* X_1 *and* X_2 *are dependent. For example, recently, Wong, et al. (2025) find that portfolios including Bitcoin could only second-order stochastically dominate other assets, and portfolios including T bills could only second-order stochastically dominate other assets, but portfolios including both Bitcoin and T bills could first-order stochastically dominate other assets.*

Post and Levy (2005) use various stochastic dominance criteria that account for (local) risk seeking to analyze market portfolio efficiency relative to benchmark portfolios formed on market capitalization, book-to-market equity ratio and price momentum. They find that reverse S-shaped utility functions with risk aversion for losses and risk seeking for gains can explain stock returns. One may ask whether the ADSD theory could be compared with the theories for investors with revise S-shaped utilities. Our answer is "Yes". If one applies Extension 6.1 and set $\omega_1 = 1$ when $u_1 \geq 0$, $\omega_1 = 0$ when $u_1 <$, $\omega_2 = 1$ when $u_2 < 0$, and $\omega_2 = 0$ when $u_1 \geq 0$, then investors with utilities as shown in Equation (6.1) are all investors with revise S-shaped utilities so that all the theories and applications for investors with revise S-shaped utilities, see, for example,

One may ask whether it is possible to use the ADSD theory but not the extension of the ADSD theory to compare with the theories of investors with revise S-shaped utilities. Our answer is YES. We construct the following example to illustrate the possibility:

Example 6.1 *Consider the Earnings per share (EPS) for Stocks A and B as shown in the following table: One can find that A second-order stochastically dominates B in the*

Table 6.1: Earnings per share (EPS) for Stocks A and B

EPS	Prob. (A)	Prob. (B)
-3	0.25	0.4
-2	0.25	0.1
-1	0	0
0	0	0
1	0.15	0.1
2	0.1	0.2
3	0.25	0.2
	1.0	1.0

sense of both ASD and MSD.[6] *Thus, both risk averters (and the u_1 of the ADSD investors) and investors with reverse S-shaped utilities will prefer to invest in A to B.*

7 Concluding Remarks

In this paper, we aim to develop a theory so that we can apply the theory to find a new solution to the paradox raised by Friedman and Savage (1948). To do so, we first modify the two-piece utility function idea proposed by Fishburn and Kochenberger (1979) and

[6] Readers may refer to Wong and Chan (2008) for the definition of MSD.

others, use the idea of proposed by Thon and Thorlund-Petersen (1988) to use two-way stochastic dominance to first define the j-order risk-averse and risk-seeking utility (We call it AD utility), and use the idea from Chew and Tan (2005) to get risk-seeking component in the utility function. We then develop a new stochastic dominance (SD) theory (We call it ADSD theory) for investors with AD utility that consists of both risk-averse and risk-seeking components. Based on the ADSD theory we developed in our paper, we not only find a new solution to the paradox raised by Friedman and Savage that investors with AD utility (We call them AD investors/subjects/individuals) could buy both insurance and try their luck with lotteries get higher expected utility, but also find that AD subjects could invest in both completely diversified portfolio and individual assets or, in general, buy any pair of both less-risky and riskier assets.

Readers may refer to Section 6 for our discussion of the weaknesses and the extensions of the theory developed in this paper. There are many other applications of the extension of the ADSD theory. For example, when investors believe that there is a buy signal (by using any tool, for example, technical analysis, see, for example, See, for example, Wong, *et al.* (2001, 2003)), then one could become a risk seeker, and when investors believe that there is a sell signal (by using any tool, for example, technical analysis), then one could become a risk averter. Thus, one could apply Extension 6.1 in which both ω_1 and ω_2 vary in time to explain this phenomenon. We leave the extension and other applications for future studies.

The theory developed in our paper could only be used to find that investors could invest in two assets, for example, bonds and stocks (Wong, *et al.,* 200; Bouri, *et al.,* 2018; Chow, *et al.,* 2019), bonds and futures, and stocks and futures (Lean, *et al.,* 2010, 2015; Qiao, *et al.,* 2012, 2013; Clark, *et al.,* 2016), but not in more than two assets, for example, investing in bonds, stocks, and futures. Thus, the extension could include developing a theory to explain why investors could invest in more than two assets with different risks. The extension could also include developing a theory to explain investors' behaviors in other assets, for example, housing (Qiao and Wong, 2015; Tsang, *et al.,* 2016), other areas, for example, production (Egozcue, *et al.,* 2015; Guo, *et al.,* 2015, 2018, 2020; Guo and Wong, 2019), banking and international trade (Broll, *et al.,* 2006, 2015), and different market situations (Vieito, *et al.,* 2015; McAleer, *et al.,* 2016; Zhu, *et al.,* 2019).

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